Fixing My Proof of 8.13:

1.) Recall that if $\sum |c_n[f]| < \infty$,

$$|f(x) - S_n[f](x)| \leq \sum_{|k|>n} |c_n[f]| \to 0 \quad \text{as } n \to \infty$$

regardless of x.

2.) Recall that if $\sum |k|^m |c_n[f]| < \infty$ for $k \in \mathbb{N}_0$, $f \in C^m(\mathbb{T})$. ~~In particular, consider~~

~~[illegible]~~

3.) If $\sum |k|^m c_k < \infty$ for all $m \in \mathbb{N}_0$, $c_k \to 0$. Thus, $\{c_k\}$ is bdd. and

$$\sum |c_k|^2 \leq |c_0| + |c_1| + |c_2| + \sum k^m |c_k|$$

for some large m, so $\sum |c_k|^2 < \infty$ and $g(x) = \sum c_k e^{ikx}$ is an L^2 ($\&$ C^∞) function.

$$u(t,x) = \sum_{k\in\mathbb{Z}} c_k[h]\, e^{-k^2 t} e^{ikx}$$

& $\sum_{k\in\mathbb{Z}} |c_k[h]|^2 < \infty \Rightarrow$ $|c_k[h]|$ bdd.

For fixed $t>0$,

$$\sum_{k\in\mathbb{Z}} k^m e^{-k^2 t} \le \int_{\mathbb{R}} x^m e^{-x^2 t}\,dx$$

$< \infty$ by integration-by-parts.

Such that $c_k[u(t,x)] = c_k[h]\, e^{-k^2 t}$

has $\sum_{k\in\mathbb{Z}} |k^m c_k[u(t,x)]| < \infty$

and $u(t,\cdot) \in C^\infty(\mathbb{T})$. Let $u_n = \sum_{k=-n}^{n} c_k e^{-k^2 t} e^{ikx}$

Similarly, $\partial u_n/\partial t = \sum_{-n}^{n} (-k^2)\, c_k[h]\, e^{-k^2 t} e^{ikx}$

has $\sum_{-\infty}^{\infty} |(-k^2)\, c_k[h]\, e^{-k^2 t}| < \infty$

So $\lim_{n\to\infty} \partial u_n/\partial t\,(t,x) = g(t,x)$ exists $\forall\, t>0,\ x\in\mathbb{T}$.

Further, pick $\varepsilon>0$ and restrict to $t>\varepsilon$,

& $\sum_{-\infty}^{\infty} k^m |(-k^2)\, c_k[h]\, e^{-k^2 t}|$

$\le \sum_{-\infty}^{\infty} k^{m-2} |c_k[h]|\, e^{-k^2\varepsilon} < \infty$

So that, as in our prev. proof, $\partial u_n/\partial t \to g$ uniformly.

As before, we may show $\partial u/\partial t = g$.

Now, in $(\varepsilon,\infty)\times\mathbb{T}$

$$\partial u/\partial t - \partial^2 u/\partial x^2 = \lim_{n\to\infty} \partial u_n/\partial t - \partial^2 u_n/\partial x^2 \quad \text{(uniform limit in } t, x\text{)}$$

$$= \lim_{n\to\infty} 0 = 0$$

So u satisfies the heat eqn. in $(\varepsilon,\infty)\times\mathbb{T}$.

Let $\varepsilon\to 0$, & this works in $(0,\infty)\times\mathbb{T}$.

- If $h \in C'(\mathbb{T})$, $\sum_{-\infty}^{\infty} c_k[h] e^{ikx}$ converges to h uniformly, so

$$u(t,x) - h(x) = \lim_{n\to\infty} \sum_{-n}^{n} c_k[h] e^{ikx} (e^{-k^2 t} - 1)$$

and the limit is uniform.

Hence, pick N such that $\sum_{|k|>n} c_k[h](e^{-k^2 t}-1) < \varepsilon/2$ ~~for~~ $0<t<\delta$, and pick $\delta>0$ such that for $k = -n, \dots, 0, \dots, n$

$$|e^{-k^2 t} - 1| < \frac{\varepsilon}{2\|h\|_\infty}$$

and for $0<t<\delta$

$$|u(t,x) - h(x)| \le \sum_{|k|\le n} \left|c_k[h] \frac{\varepsilon}{2\|h\|_\infty} e^{ixk}\right| + \sum_{|k|>n} |c_k[h](e^{-k^2 t}-1)|$$

$$\le \varepsilon/2 + \varepsilon/2 = \varepsilon$$

& $\lim_{t\to 0} u(t,x) = h(x)$.

$\boxed{\text{Corollary}}$ Suppose $h \in C^0([0,\ell])$ and satisfies Dirichlet or Neumann Boundary Conditions. The heat equation on $[0,\infty) \times [0,\ell]$ admits a solution $u \in C^\infty((0,\infty) \times [0,\ell])$ under the same B.C. such that

$$\lim_{t \to 0} u(t,x) = h(x).$$

for each $x \in [0,\ell]$.

$\boxed{\text{Pf}}$ Extend $h(x)$ to an even, 2ℓ-periodic C^0 function on $\mathbb{R}$. Then, $\cancel{h(x) \in C^0(\mathbb{T})}$. $h\left(\frac{x\pi}{\ell}\right) \in C^0(\mathbb{T})$.

We may solve as in the previous theorem to obtain $\tilde{u}(t,x)$ a solution in $C^\infty((0,\infty) \times \mathbb{T})$ and set $u(t,x) = \tilde{u}\left(t, \frac{x\ell}{\pi}\right)$. $\square$